Econ. 511b Spring 1999 George Hall and Chris Sims

Solving Rational Expectations Models via Jordan Decompositions*

1. Introduction

These notes largely reproduce what is in the section on solving by Jordan decomposition in "Solving Linear Rational Expectations Models". The differences are that here the exposition is simpler because some hard special cases are assumed away, and serially correlated exogenous disturbances are allowed, as in the Spring 1999 lectures.

Solving by Jordan decomposition is not a sound computational strategy for general systems, because the Jordan decomposition is numerically unstable. On the other hand, for small systems it can be implemented by hand and understood intuitively more easily than methods based on numerically more robust matrix decompositions.

Our simplifying assumptions in these notes are

- All roots are distinct.
- Stability conditions restrict all variables jointly not to grow faster than some rate ϕ^{-t}. [This follows Blanchard and Kahn's classic paper, but fails to deal with some practically important issues. For example, it can easily happen that wealth is not bounded by the problem's structure, but its ratio to consumption is.]

2. Model and Assumptions

We consider models of the form

$$\underset{n\times 1}{y(t)} = c + Ay(t-1) + \Psi \underset{q\times 1}{z(t)} + \Pi \underset{k\times 1}{\eta(t)}\,, \tag{1}$$

where $z(t)$ is an exogenously given stochastic process and $\eta(t)$ is an endogenous error term satisfying $E_t\eta(t+1) = 0$ for all t. We assume that a solution must satisfy $E[y(t+s)\phi^s] \underset{t\to\infty}{\to} 0$ for some given $\phi \in (0,1)$.

The matrix A will have a Jordan decomposition $A = P\Lambda P^{-1}$, with Λ diagonal. (Of course if we were not assuming that no roots repeat, Λ would not necessarily be diagonal.) Therefore premultiplying (1)by P^{-1} and defining $w(t) = P^{-1}y(t)$, we have

$$w(t) = P^{-1}c + \Lambda w(t-1) + P^{-1}\Psi z(t) + P^{-1}\Pi\eta(t)\,. \tag{2}$$

Let λ_i be the i't diagonal element of Λ, and sort them (and correspondingly the rows of P^{-1} and columns of P) so that the λ_i's that exceed ϕ^{-1} in absolute value are all in the lower right corner of Λ.

3. Solution

Assuming that $E[z(t)]$ is bounded, we can solve the lower set of equations in (2) forward, i.e. set

$$w_2(t) = \sum_{s=1}^{T} \Lambda_2^{-s} \cdot \left(-P^{2\cdot}c - P^{2\cdot}\Psi z(t+s) - P^{2\cdot}\Pi\eta(t+s) \right) + \Lambda_2^{-T} w_2(t+T)$$

$$= -\left(\Lambda_2 - I\right)^{-1} P^{2\cdot}c + \sum_{s=1}^{\infty} -E_t \left[\Lambda_2^{-s} P^{2\cdot}\Psi z(t+s) \right] \,, \quad (3)$$

where Λ_2 is the lower right square submatrix of Λ containing all the roots that exceed ϕ^{-1} and $P^{2\cdot}$ is the corresponding rows of P^{-1}. This is justified because $\Lambda_2^{-s} w(t+s) \to 0$ as $t \to \infty$ in any solution, by assumption, and also because, since $w_2(t)$ is known at t, taking E_t of the middle term in the two equalities in (3) does not change it. By the same token, applying the operator $E_{t+1} - E_t$ to that middle term does not change it, which allows us to conclude that

$$\sum_{s=1}^{\infty} \Lambda_2^{-s} P^{2\cdot}\Psi (E_{t+1} - E_t) z(t+s) + \Lambda_2^{-1} P^{2\cdot}\Pi\eta(t+1) = 0 \,. \quad (4)$$

This equation may or may not have a solution in $\eta(t)$ for every possible value of the terms in z. (Because z is exogenous, we assume the terms in z can take on any form, i.e. satisfy no linear restrictions. If they do satisfy such restrictions, the model should be reformulated so those restrictions become part of the model.) It necessarily has such a solution whenever $P^{2\cdot}\Pi$ is of full column rank. Thus if the number m of unstable roots (rows in $P^{2\cdot}$) is less than or equal to the number k of endogenous error terms (columns in Π), we can expect generally that a solution will exist. The exception would arise when there are enough exact linear relations among the columns of $P^{2\cdot}\Pi$ so that every $m \times m$ submatrix of it is singular. When $k < m$, existence requires that each column of $P^{2\cdot}\Psi$ lie in the k-dimensional subspace of $\mathbb{R}^m$ spanned by the columns of $P^{2\cdot}\Pi$. This requires "unlikely" or "knife-edge-case" exact linear relationships within the model, though of course in any particular model it could indeed happen that such exact relationships exist. If we assume the "regularity conditions" that the matrix $P^{2\cdot}[\Psi\Pi]$ is of full row rank m and $P^{2\cdot}\Pi$ is of full rank $\min(m,k)$, then we get the "generic" conclusion that a solution exists when $m \leq k$ and does not exist when $m > k$.

This leads us to the first main conclusion:

Existence: A solution to the system (1) that contains no component growing as fast as ϕ^{-t} exists **if and only if** the columns of $P^{2\cdot}\Pi$ span the space generated by the columns of $P^{2\cdot}\Psi$. A **sufficient** condition for this is that $P^{2\cdot}\Pi$ be of full column rank. **Generically**, we expect a solution will exist when $k \geq m$, that is when the number of endogenous error terms equals or exceeds the number of unstable roots, and will not exist when $k < m$.

For the solution to be unique, given that it exists, it must be true that the value of $P^{2\cdot}\Pi\eta(t)$ that is determined in the forward-solved block of (2) also determines the value of $P^{1\cdot}\Pi\eta(t)$ that appears in the stable block of (2). If $P^{-1}\Pi$ is of full column rank k, then we get uniqueness only if $P^{2\cdot}\Pi$ is of rank k. Thus under the regularity conditions that $P^{2\cdot}\Pi$ is of full rank $\min(m,k)$ and $P^{-1}\Pi$ is of rank k, we get the "generic" conclusion that uniqueness holds if and only if $m \geq k$. This leads us to the second main conclusion:

Uniqueness: If a solution to the system (1) that contains no component growing as fast as ϕ^{-t} exists, the solution is unique **if and only if** the rows of $P^{2\cdot}\Pi$ span the space generated by the rows of $P^{1\cdot}\Pi$. A **sufficient** condition for this is that $P^{2\cdot}\Pi$ be of full row rank. **Generically**, we expect the solution to be unique when the number of endogenous error terms is less than or equal to the number of unstable roots, and otherwise not to be unique.

The generic criterion for both existence and uniqueness, then, is that the number of unstable roots exactly match the number of endogenous error terms. When there are "too many" unstable roots, existence fails. When there are "too few", uniqueness fails.

4. Relation to Blanchard-Kahn Paper

The classic paper by Blanchard and Kahn, "The Solution of Linear Difference Models under Rational Expectations", *Econometrica* 48, 1980, 1305-1311, uses a similar approach. It differs in that it assumes the regularity conditions required to justify the generic root-counting conditions and does not discuss explicitly what happens when they are violated. Also, Blanchard and Kahn assume that each endogenous error term appears in just one equation, in fact that in our notation

$$\Pi = \begin{bmatrix} 0 \\ I \end{bmatrix} ,$$

so that each endogenous error corresponds to one element of y. The variables then divide neatly into "predetermined" and "non-predetermined" categories. It is not always easy to make this connection between endogenous errors and non-predetermined variables. The formulation in these notes is more general, and thus more widely applicable.